Free random Lévy and Wigner-Lévy matrices

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(Received 30 January 2007; published 30 May 2007)

We compare eigenvalue densities of Wigner random matrices whose elements are independent identically distributed random numbers with a Lévy distribution and maximally random matrices with a rotationally invariant measure exhibiting a power law spectrum given by stable laws of free random variables. We compute the eigenvalue density of Wigner-Lévy matrices using (and correcting) the method by Bouchaud and Cizeau, and of free random Lévy (FRL) rotationally invariant matrices by adapting results of free probability calculus. We compare the two types of eigenvalue spectra. Both ensembles are spectrally stable with respect to the matrix addition. The discussed ensemble of FRL matrices is maximally random in the sense that it maximizes Shannon's entropy. We find a perfect agreement between the numerically sampled spectra and the analytical results already for matrices of dimension $N=100$. The numerical spectra show very weak dependence on the matrix size N as can be noticed by comparing spectra for $N=400$. After a pertinent rescaling, spectra of Wigner-Lévy matrices and of symmetric FRL matrices have the same tail behavior. As we discuss towards the end of the paper the correlations of large eigenvalues in the two ensembles are, however, different. We illustrate the relation between the two types of stability and show that the addition of many randomly rotated Wigner-Lévy matrices leads by a matrix central limit theorem to FRL spectra, providing an explicit realization of the maximal randomness principle.

DOI: 10.1103/PhysRevE.75.051126 PACS number(s): 02.50.−r, 02.60.−x, 89.90.+n

I. INTRODUCTION

Applications of random matrix theory cover many branches of physics and cross-disciplinary fields [1] involving multivariate analysis of large and noisy data sets [2]. The standard random matrix formulation belongs to the Gaussian basin, with a measure that is Gaussian or polynomial with finite second moment. The ensuing macroscopic spectral distribution is localized with finite supports on the real axis. The canonical distribution for a Gaussian measure is Wigner's semicircle.

The class of stable (Lévy) distributions [3] is, however, much larger (the Gaussian class represents only one fixed point in the stability basin of the Lévy class), and one is tempted to ask why the theory of random Lévy matrices is not so well-established. The case of Lévy randomness is far from being academic, and many distributions in physics and outside (finance, networks) exhibit powerlike behavior referred to as *fat* or *heavy tails* [4].

One of the chief reasons for why the theory of random Lévy matrices is not yet well-understood is the technical difficulty inherent to these distributions. First, even for one-dimensional stable distributions, the explicit form of the probability distribution functions (PDFs) is known analytically only in a few cases [5,6]. Second, Lévy distributions have divergent moments, and the finiteness of the second moment (condition for the Gaussian stability class) is usually a key for many of the techniques established in random matrix theory. Third, numerical studies involving powerlike behavior on infinite supports require enormous statistics and are very sensitive to systematic errors.

In 1994, Bouchaud and Cizeau [7] (BC) considered large $N\times N$ random symmetric matrices with entries sampled from one-dimensional, stable distributions. In the large N limit they obtained analytical equations for the entries of the resolvent, and then checked their predictions for the spectra using numerically generated spectra by random sampling. The agreement was fair, although not perfect. Contrary to the standard Gaussian-like ensembles, the measure in the BC approach was not rotationally invariant.

In 2002, following the work in [8], we suggested another Lévy-type ensemble [9] [hereafter free random Lévy (FRL)]. By construction its measure is rotationally invariant. The average spectral distribution in this ensemble is stable under the matrix convolution of two independent but identical ensembles. It is similar to the stability property of one-dimensional Lévy distributions. The measure is nonanalytic in the matrix H and universal at large H with a potential $V(H)\approx \ln H^2$. This weak logarithmic rise in the asymptotic potential is at the origin of the long tail in the eigenvalue spectra.

The present work will compare the Wigner-Lévy (WL) and FRL results as advertised in [10]. In Sec. II we reanalyze and correct the original arguments for the resolvent presented in [7]. Our integral equations for the resolvent and spectral density are different from the ones in [7]. We carry explicit analytical transformations and expansions to provide insights to the spectrum. We show that there is a perfect agreement between the analytical and numerical results obtained by sampling large Lévy matrices. We also discuss the relation of ours and BC's results. In Sec. III we recall the key concepts behind FRL ensembles. In large N the resolvent obeys a simple analytic equation. The resulting spectra are compared to the spectra following from the corrected BC analysis. The WL and FRL matrices represent two types of stability under matrix convolution. In both cases we have a power behavior in the tails of the spectrum. By a pertinent rescaling we may in fact enforce the same tail behavior and compare the spectra. The observed differences disappear in the Gauss limit and become more pronounced in the Cauchy limit. In Sec. IV we explain the relation between the two types of stability on a simple example: we construct sums of WL matrices rotated by random $O(N)$ matrices and show that the spectrum of these sums converges by a matrix central limit theorem to the pertinent symmetric FRL spectrum. Our conclusions are in Sec. V.

II. WIGNER-LÉVY MATRICES

A. Definition of ensemble

In a pioneering study on random Lévy matrices, Bouchaud and Cizeau [7] discussed a Wigner ensemble of $N\times N$ real symmetric random matrices with elements being independent identically distributed (IID) random variables: with probability density function following a Lévy distribution $P(x)\equiv N^{1/\mu}L_\mu^{C,\beta}(N^{1/\mu}x)$, with μ being the stability index, β the asymmetry parameter, and C the range of the distribution (see below). We shall call these matrices Wigner-Lévy (WL) or Bouchaud-Cizeau (BC) matrices. The probability measure for the ensemble of such matrices is given by

$$d\mu_{WL}(H)=\prod_{i\leq j}P(H_{ij})dH_{ij}. \tag{1}$$

The scaling factor $N^{1/\mu}$ in the PDF makes the limiting eigenvalue density independent of the matrix size N when $N\to\infty$. Alternatively one can think of the matrix elements H_{ij} as if they were calculated as $H_{ij}=h_{ij}/N^{1/\mu}$ with h_{ij} being IID random numbers independent of N: $p(x)=L_\mu^{C,\beta}(x)$.

Lévy distributions are notoriously hard to write explicitly (except in a few cases), but their characteristic functions are more user friendly [5]

$$L_\mu^{C,\beta}(x)=\frac{1}{2\pi}\int dk\hat{L}(k)e^{ikx}, \tag{2}$$

where the characteristic function is given by

$$\log\hat{L}(k)=-C|k|^\mu[1+i\beta\,\mathrm{sgn}(k)\tan(\pi\mu/2)]. \tag{3}$$

The parameters μ, β, and C are related to the asymptotic behavior of $L_\mu^{C,\beta}(x)$

$$\lim_{x\to\pm\infty}L_\mu^{C,\beta}(x)=\gamma(\mu)\frac{C(1\pm\beta)}{|x|^{\mu+1}} \tag{4}$$

with the μ-dependent parameter $\gamma(\mu)$ given by

$$\gamma(\mu)=\Gamma(1+\mu)\sin\left(\frac{\pi\mu}{2}\right). \tag{5}$$

Here μ is the stability index defined in the interval (0, 2], $-1\leq\beta\leq1$ measures the asymmetry of the distribution, and the range $C>0$ is the analog of the variance, in a sense that a typical value of x is $C^{1/\mu}$. A standard choice corresponds to $C=1$.

We shall consider here only the stability index in the range (1, 2), although as will be shown later, results obtained in this range seem to be valid also for $\mu=1$. We also assume that all random variables have zero mean.

B. Determination of eigenvalue density

A method of calculating the eigenvalue density of the Wigner-Lévy matrices was invented by Bouchaud and Cizeau [7]. Let us in this section briefly recall the main steps of the method. It is convenient to introduce the resolvent, also called the Green's function:

$$g(z)=\frac{1}{N}\langle\mathrm{Tr}\,G(z)\rangle, \tag{6}$$

where elements of the matrix $G(z)$ are

$$G_{ij}(z)=(z-H)_{ij}^{-1} \tag{7}$$

and the averaging is carried out using the measure (1). The resolvent contains the same information as the eigenvalue density $\rho(\lambda)$. Indeed if one approaches the real axis one finds that $\rho(\lambda)=-1/\pi\lim_{\epsilon\to0^+}\mathrm{Im}\,g(\lambda+i\epsilon)$. This is how one usually calculates $\rho(\lambda)$ from $g(z)$. For the Wigner-Lévy ensemble, where individual matrix elements have large scale-free statistical fluctuations, a slightly different method turns out to be more practical—a method which allows one to avoid problems in taking the double limit (first $N\to\infty$ and $\epsilon\to0^+$) in which the fluctuations are suppressed in an uncontrollable way in the presence of an imaginary part of z. In the BC method z is kept on the real axis and fluctuations are not suppressed, so one can safely take the large N limit.

The method goes as follows [7]. One first generates a symmetric $N\times N$ random matrix H using the measure (1) and then by inverting $H-z$ one calculates the resolvent $G(z)$ [Eq. (7)]. Next one adds a new row (and a symmetric column) of independent numbers identically distributed as those in the old matrix H. One obtains a new $(N+1)\times(N+1)$ matrix H^{+1}, where +1 emphasizes that it has one more row and one more column than H. It is convenient to number elements of the original matrix H_{ij} by indices running over the range $i,j=1,\ldots,N$, and assign the index 0 to the new row and the new column so that now indices of H^{+1} run over the range $0,\ldots,N$. If one inverts the matrix $H^{+1}-z$ one obtains a new $(N+1)\times(N+1)$ resolvent $G^{+1}(z)$. One can show that it obeys a recursive relation [7]

$$z-\frac{1}{G_{00}^{+1}(z)}=H_{00}+\sum_{i,j}^{N}H_{0i}H_{0j}G_{ij}(z)=\frac{h_{00}}{N^{1/\mu}}+\sum_{i,j}^{N}\frac{h_{0i}h_{0j}G_{ij}(z)}{N^{2/\mu}}, \tag{8}$$

which relates the element $G_{00}^{+1}(z)$ of the $(N+1)\times(N+1)$ resolvent to the elements of the old $N\times N$ resolvent $G(z)$. One can also derive similar equations for off-diagonal elements of $G^{+1}(z)$

$$\frac{G_{0j}^{+1}(z)}{G_{00}^{+1}(z)} = \sum_i^N \frac{h_{0i}G_{ij}(z)}{N^{2/\mu}}. \tag{9}$$

The difference between the probability distribution of the elements of $G^{+1}(z)$ and of $G(z)$ disappears in the limit $N\to\infty$. The diagonal elements of the matrix $G^{+1}(z)$ are identically distributed as the diagonal elements of the matrix $G(z)$. The same holds for off-diagonal ones. Moreover, in this limit all elements of the G matrix become independent of each other. In particular all diagonal elements of $G(z)$ become independent identically distributed (IID) random variables as $N\to\infty$. One can use Eqs. (8) and (9) to derive self-consistency equations for the probability density function (PDF) for diagonal and the PDF for off-diagonal elements. We are here primarily interested in the distribution of the diagonal elements, as we shall see below. The self-consistency equation for the probability distribution of diagonal elements follows from Eq. (8) and is independent of the distribution of the off-diagonal elements. This can be seen as follows. Let us first define after Bouchaud and Cizeau [7] a quantity:

$$S_0(z) = z - \frac{1}{G_{00}^{+1}(z)}. \tag{10}$$

It is merely a convenient change of variables suited to the left-hand side of Eq. (8). It is clear that if one determines the PDF for $S=S_0(z)$, one will also be able to determine the PDF for $G=G_{00}^{+1}(z)$ since the two PDFs can be obtained from each other by the change of variables (10):

$$P_G(G) = \frac{1}{G^2}P_S\left(z - \frac{1}{G}\right), \tag{11}$$

where P_G and P_S are PDFs for $G_{00}^{+1}(z)$ and $S_0(z)$, respectively. Since the probability distribution P_G is identical for all diagonal elements of G, it remains to determine the probability distribution P_S for the quantity $S_0(z)$.

Let us sketch how to do that. First observe that the first term in Eq. (8) can be neglected at large N, so the equation assumes the form

$$S_0(z) = \sum_i^N \frac{h_{0i}^2 G_{ii}(z)}{N^{2/\mu}} + \sum_{i\neq j}\frac{h_{0i}h_{0j}G_{ij}(z)}{N^{2/\mu}}. \tag{12}$$

Now observe that by construction h_{0i} are independent of $G_{ij}(z)$. We shall now show that for large N the first term in Eq. (12) dominates over the second one. We shall modify here the argument used in [7], where it was assumed that the off-diagonal elements $G_{ij}N(z), i\neq j$ are suppressed by a factor $1/N^{1/\mu}$ with respect to the diagonal elements. Instead we note that by construction the quantities $G_{ij}(z)$ are statistically independent of $h_{0i}, i=0,\ldots,N$. Since we shall only be interested by the diagonal elements of the resolvent matrix, we may replace the contribution of the off-diagonal elements $h_{0i}h_{0j}G_{ij}(z), i\neq j$ by their averaged values. As a result, the contribution of the off-diagonal terms averages out. Note that this is also true in the Gaussian limit $\mu=2$ where following [7] we may replace all elements of Eq. (12) by their respective averages. Taking this into account and omitting the subleading contribution H_{00}, we get

$$S_0(z) = \sum_i^N \frac{h_{0i}^2 G_{ii}(z)}{N^{2/\mu}}. \tag{13}$$

Thus the problem was simplified to an equation where the left-hand side ($S_0=z-1/G_{00}^{+1}$) and the right-hand side depend only on the diagonal elements of the G-matrix, which as we mentioned before, are identically distributed in the limit $N\to\infty$. Using Eq. (13) one can derive a self-consistency equation for the probability density function (PDF) P_G for diagonal elements of the matrix G.

C. Generalized central limit theorem

To proceed further we apply with Bouchaud and Cizeau [7] the *generalized central limit theorem* to derive the universal behavior of the sum on the right-hand side of Eq. (13) in the limit $N\to\infty$:

(1) If the h_{0i}'s are sampled from the Lévy distribution $L_\mu^{C,\beta}$, the squares $t_i=h_{0i}^2$ for large t_i are distributed solely along the positive real axis, with a heavy tail distribution:

$$\propto \gamma(\mu)\frac{C dt_i}{t_i^{1+\mu/2}} \tag{14}$$

irrespective of β. The sum

$$\sum_i^N \frac{t_i}{N^{2/\mu}} \tag{15}$$

is distributed following $L_{\mu/2}^{C',1}(t_i)$. The range parameters C and C' are related by Eq. (3),

$$2C'\gamma(\mu/2) = C\gamma(\mu). \tag{16}$$

The factor 2 on the left-hand side corresponds to the sum $(1+\beta)+(1-\beta)$ appearing as a contribution from positive and negative values of the original distribution of h_{0i}. This relation is important when comparing to the numerical results below where $C=1$ is used. From now on (and to simplify the equations) we assume instead that $C'=1$.

(2) By virtue of the central limit theorem the following sum

$$\sum_i^N \frac{G_{ii}(z)t_i}{N^{2/\mu}} \tag{17}$$

of IID heavy tailed numbers t_i is for $G_{ii}(z)=O(N^0)$ and $N\to\infty$ Lévy distributed with the PDF: $L_{\mu/2}^{C(z),\beta(z)}$, which has the stability index $\mu/2$ and the effective range $C(z)$ and the asymmetry parameter $\beta(z)$ calculated from the equations

$$C(z) = \frac{1}{N}\sum_i^N |G_{ii}(z)|^{\mu/2} \tag{18}$$

and

$$\beta(z) = \frac{\frac{1}{N}\sum_i^N |G_{ii}(z)|^{\mu/2}\,\mathrm{sgn}[G_{ii}(z)]}{\frac{1}{N}\sum_i^N |G_{ii}(z)|^{\mu/2}}, \tag{19}$$

which follow from the composition rules for the tail amplitudes of IID heavy tailed numbers t_i defined above.

D. Integral equations

We saw in the previous section that the generalized central limit theorem implies for large N that the "self-energy" $S=S_0(z)$ is distributed according to the Lévy law $P_S(S)=L_{\mu/2}^{C(z),\beta(z)}(S)$ with the stability index $\mu/2$ being one-half of the stability index of the Lévy law governing the distribution of individual elements of the matrix H, and with the effective range parameter $C(z)$ and the asymmetry parameter $\beta(z)$ which can be calculated from Eqs. (18) and (19), respectively. One should note that the effective parameters $C(z),\beta(z)$ of the distribution $P_S(S)$ are calculated for $C=1$ and that they are independent of β of the probability distribution: $L_\mu^{C,\beta}(h_{ij})$ of the H-matrix elements.

The sums on the right hand side of Eqs. (18) and (19) for $C(z)$ and $\beta(z)$ have a common form $\frac{1}{N}\Sigma_i f(G_{ii}(z))$. Since in the limit $N\to\infty$, the diagonal elements become IID, the sums can be substituted by integrals over the probability density for G_{ii}:

$$\frac{1}{N}\sum_i^N \langle f(G_{ii}(z))\rangle = \int dG P_G(G) f(G) = \int \frac{dG}{G^2} P_S\left(z-\frac{1}{G}\right) f(G), \tag{20}$$

where in the second step we used Eq. (11). Since the distribution $P_S(S)=L_{\mu/2}^{C(z),\beta(z)}(S)$ is known up to the values of two effective parameters $C(z)$ and $\beta(z)$ Eqs. (18) and (19) can be written as self-consistency relations for $\beta(z)$ and $C(z)$:

$$C(z) = \fint_{-\infty}^{\infty} \frac{dG}{G^2}|G|^{\mu/2} L_{\mu/2}^{C(z),\beta(z)}(z-1/G),$$

$$\beta(z) = \frac{\fint_{-\infty}^{\infty} \frac{dG}{G^2}|G|^{\mu/2}\,\mathrm{sgn}(G) L_{\mu/2}^{C(z),\beta(z)}(z-1/G)}{\fint_{-\infty}^{\infty} \frac{dG}{G^2}|G|^{\mu/2} L_{\mu/2}^{C(z),\beta(z)}(z-1/G)}. \tag{21}$$

The symbol $\fint$ stands for principal value of the integral. Notice here the difference between our second equation and that in [7]. In addition to [7] we also note that the resolvent takes the form

$$g(z) = \frac{1}{N}\sum_i G_{ii}(z) \to g(z) = \fint_{-\infty}^{\infty} \frac{dG}{G^2} G L_{\mu/2}^{C(z),\beta(z)}(z-1/G). \tag{22}$$

The integrals (21) and (22) can be rewritten using the new integration variable $x=1/G$ as

$$C(z) = \int_{-\infty}^{+\infty} dx|x|^{-\mu/2} L_{\mu/2}^{C(z),\beta(z)}(z-x),$$

$$\beta(z) = \frac{\int_{-\infty}^{+\infty} dx\,\mathrm{sgn}(x)|x|^{-\mu/2} L_{\mu/2}^{C(z),\beta(z)}(z-x)}{\int_{-\infty}^{+\infty} dx|x|^{-\mu/2} L_{\mu/2}^{C(z),\beta(z)}(z-x)}, \tag{23}$$

and

$$g(z) = \fint_{-\infty}^{\infty} \frac{dx}{x} L_{\mu/2}^{C(z),\beta(z)}(z-x). \tag{24}$$

All the steps above require both z and $G_{ii}(z)$ to be strictly real. The argument cannot be extended to the complex z plane. So all integrals above should be interpreted as principal value integrals, wherever it is necessary. Since $\mu<2$ all integrals in Eq. (23) are convergent also in the usual sense.

The equation for $g(z)$ can be rewritten as

$$g(z) = \fint_{-\infty}^{\infty} \frac{dx}{z-x} L_{\mu/2}^{C(z),\beta(z)}(x). \tag{25}$$

Notice the nontrivial dependence on z in the parameters of the Lévy distribution. The above equation can be a source of confusion, since its structure resembles another representation of $g(z)$,

$$g(z) = \fint_{-\infty}^{\infty} \frac{d\lambda}{z-\lambda}\rho(\lambda), \tag{26}$$

which superficially looks as if one could identify in Eq. (25) x and λ and $L_{\mu/2}^{C(z),\beta(z)}(x)$ with $\rho(\lambda)$. This is not the case, and one should instead invert Eq. (25) using the inverse Hilbert transform:

$$\rho(\lambda) = \frac{1}{\pi^2}\fint_{-\infty}^{\infty} \frac{dz}{z-\lambda} g(z). \tag{27}$$

In other words, one first has to reconstruct numerically the real part of the resolvent, and only then compute numerically the spectral function $\rho(\lambda)$, using the "dispersive relation" (27). This is a difficult and rather subtle procedure. In the next section we give some analytical insights to the integral equations that would help solve them and extract the spectral function.

E. Analytical properties useful for numerics

One cannot do the integrals from the previous section analytically. As mentioned, one cannot even write down an explicit form of the Lévy distribution. It is a great numerical challenge to solve the problem numerically even if all the expressions are given. One realizes that already when one tries to compute the Fourier integral (2) of the characteristic function since one immediately sees that the integrand in the form (2) is a strongly oscillating function making the numerics unstable. Fortunately using the power of the complex analysis one can change this integral to a form which is numerically stable. So in this section we present some analytic tricks which allow one to reduce the problem of computing the eigenvalue density as formulated in the previous section to a form which is well-suited to the numerical computation.

To simplify Eq. (23) we proceed in steps. First, we make use of the Lévy distribution through its characteristic

$$L_{\mu/2}^{C,\beta}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx} e^{-C|k|^{\mu/2}[1+i\zeta\,\mathrm{sgn}(k)]}, \tag{28}$$

with $\zeta=\beta\tan(\pi\mu/4)$. By rescaling through

$$k = C^{-2/\mu}k',$$
$$x = C^{2/\mu}x',$$
$$z = C^{2/\mu}z', \tag{29}$$

we can factor out the range $L_{\mu/2}^{C,\beta}(x)=C^{-2/\mu}L_{\mu/2}^{1,\beta}(x')$. Second, we make use of the following integrals:

$$\int_{-\infty}^{\infty}\frac{dx}{z-x}e^{ikx} = -2ie^{ikz}\,\mathrm{sgn}\,k,$$

$$\int_0^\infty dx'\frac{\cos(k'x')}{x'^{\mu/2}} = |k'|^{\mu/2-1}\Gamma(1-\mu/2)\sin(\pi\mu/4),$$

$$\int_0^\infty dx'\frac{\sin(k'x')}{x'^{\mu/2}} = |k'|^{\mu/2-1}\,\mathrm{sgn}(k')\Gamma(1-\mu/2)\cos(\pi\mu/4). \tag{30}$$

Last, we make use of the change of variables $p=k'^{\mu/2}$. With this in mind, we obtain

$$C^2(z') = \frac{4}{\pi\mu}\Gamma\left(1-\frac{\mu}{2}\right)\sin\left(\frac{\pi\mu}{4}\right) \times\int_0^\infty dp\,\cos[p^{2/\mu}z' - \zeta(z')p]e^{-p}, \tag{31}$$

$$\zeta(z') = \frac{\int_0^\infty dp\,\sin[p^{2/\mu}z' - \zeta(z')p]e^{-p}}{\int_0^\infty dp\,\cos[p^{2/\mu}z' - \zeta(z')p]e^{-p}}, \tag{32}$$

with $\zeta(z')=\tan(\pi\mu/4)\beta(z')$. For every z' we can iteratively solve the equation for $\zeta(z')$, then we determine $C(z')$ and use $z=C^{2/\mu}(z')z'$ to express everything in terms of z. These transformations solve Eq. (23). Using the same method we rewrite the equation for $g(z)$ as

$$\bar{g}(z') = C(z')^{2/\mu}g(z) = \frac{2}{\mu}\int_0^\infty dp p^{(2-\mu)/\mu}\sin[p^{2/\mu}z' - p\zeta(z')]e^{-p}. \tag{33}$$

These integral forms are useful to study the small-z' limit. In this case $\zeta(z')$ is an antisymmetric function of z' and has an expansion in powers of z'. Using $\zeta(z')=k_1z'+O(z'^3)$ we can recursively obtain the coefficients of this expansion. The first term is $k_1=\Gamma(1+2/\mu)/2$. Similarly $C(z')$ is a symmetric function in z',

$$C^2(z') = \frac{4}{\pi\mu}\Gamma\left(1-\frac{\mu}{2}\right)\sin\left(\frac{\pi\mu}{4}\right) + O(z'^2). \tag{34}$$

For $|z'|$ large ($z'\to\pm\infty$) a different approach is needed. In this case we follow Nolan [11] and treat the two integrals (numerator and denominator) of Eq. (32) together, i.e., we consider the integral

$$\int_0^\infty dp e^{-h(p)}, \tag{35}$$

where

$$h(p) = (1-i\zeta)p + iz'p^{2/\mu}. \tag{36}$$

Nolan's idea is to close the contour of integration in the complex p plane in the following way: at $p\to\infty$ we add an arc and afterwards continue until $p=0$ along the line where $\mathrm{Im}\,h(p)=0$. Using the parametrization $p=re^{i\theta}$ we get the parametric equation for $r(\theta)$ along this line, valid for $z'>0$

$$r(\theta) = \left(\frac{\sin(\theta_0-\theta)}{z'\cos(\theta_0)\cos\left(\frac{2}{\mu}\theta\right)}\right)^{\mu/(2-\mu)}. \tag{37}$$

The angle θ for the curve we need ($\mu\geqslant 1$) is bounded between $\theta_0=\arctan[\zeta(z')]$, where $r=0$ and $-\theta_1=-\pi\mu/4$, where $\cos(\frac{2}{\mu}\theta)$ is zero. Let us now introduce a new variable ψ through $\theta=\theta_0-\psi$ and $0\leqslant\psi\leqslant\theta_0+\theta_1$. In this range we have

$$V(\psi) = \left(\frac{\sin(\psi)}{\cos(\theta_0)\cos\left[\frac{2}{\mu}(\psi-\theta_0)\right]}\right)^{\mu/(2-\mu)},$$

$$\mathrm{Re}\,h(\psi) = \left(\frac{1}{z'}\right)^{\mu/(2-\mu)}V(\psi)\frac{\cos\left(\frac{2-\mu}{\mu}\psi - \frac{2}{\mu}\theta_0\right)}{\cos(\theta_0)\cos\left[\frac{2}{\mu}(\psi-\theta_0)\right]}. \tag{38}$$

After some manipulations we obtain

$$\int \sin[p^{2/\mu}z' - \zeta(z')p]e^{-p} = \int_0^{\theta_0+\theta_1} d\psi\left(\frac{1}{z'}\right)^{\mu/(2-\mu)}V(\psi)e^{-\mathrm{Re}\,h(\psi)} \times\left(\frac{2}{2-\mu}\frac{\cos\left[\frac{2-\mu}{\mu}(\psi-\theta_0)\right]}{\cos\left[\frac{2}{\mu}(\psi-\theta_0)\right]} - \frac{\mu}{2-\mu}\frac{\sin\theta_0}{\sin\psi}\right),$$

$$\int \cos[p^{2/\mu}z' - \zeta(z')p]e^{-p} = \int_0^{\theta_0+\theta_1} d\psi \left(\frac{1}{z'}\right)^{\mu/(2-\mu)} V(\psi)e^{-\mathrm{Re}\, h(\psi)} \times \left(\frac{2}{2-\mu} \frac{\sin\left[\frac{2-\mu}{\mu}(\psi-\theta_0)\right]}{\cos\left[\frac{2}{\mu}(\psi-\theta_0)\right]} + \frac{\mu}{2-\mu}\frac{\cos\theta_0}{\sin\psi} \right). \tag{39}$$

The resulting integrals look complicated, however, they contain both the small-z' and the large-z' asymptotics. For $z' \to 0$ we have $\psi = z' p^{2/\mu - 1}$, which reproduces the small-z' expansion presented above. For $z' \to \infty$ we have $\psi = \theta_0 + \theta_1 - u/z'^{\mu/2}$. Note that in this limit

$$\cos\left(\frac{2}{\mu}(\psi-\theta_0)\right) = \sin\left(\frac{2}{\mu}\frac{u}{z'^{\mu/2}}\right) \tag{40}$$

and the z' dependence in $\mathrm{Re}\, h(\psi)$ vanishes in leading order.

The large-z' asymptotics require some work. For the leading orders we have

$$\int \sin[p^{2/\mu}z' - \zeta(z')p]e^{-p} \approx \Gamma(1+\mu/2)\sin\theta_1 (z')^{-\mu/2},$$

$$\int \cos[p^{2/\mu}z' - \zeta(z')p]e^{-p} \approx \Gamma(1+\mu/2)\cos\theta_1 (z')^{-\mu/2}. \tag{41}$$

Thus for $z' \to \infty$ we have

$$\zeta(z') = \tan(\theta_1) + O(1/z'^{\mu/2}) \tag{42}$$

and

$$C(z') = z'^{-\mu/4}[1 + O(1/z'^{\mu/2})],$$

$$z = \sqrt{z'}[1 + O(1/z'^{\mu/2})]. \tag{43}$$

All the formulas above apply to the case $z' > 0$. One can also derive similar formulas for $z' < 0$, it is, however, more practical to use the symmetry properties of the functions $C(z')$ and $\zeta(z')$.

As a final check let us compute $\bar{g}(z')$. A rerun of the above transformations on the integrals give

$$\bar{g}(z') = \frac{2}{2-\mu}\int_0^{\theta_0+\theta_1} d\psi \left(\frac{1}{z'}\right)^{2/(2-\mu)} V(\psi)^{2/\mu} e^{-\mathrm{Re}\, h(\psi)} \times \left(\frac{2}{\mu} \frac{1}{\cos\left[\frac{2}{\mu}(\psi-\theta_0)\right]} + \frac{\sin\left(\frac{2-\mu}{\mu}\psi - \frac{2}{\mu}\theta_0\right)}{\sin\psi} \right). \tag{44}$$

Notice that both asymptotics follow from this representation. For $z' \to \infty$ we have $\bar{g}(z') = 1/z' + \cdots$, which implies $g(z) = 1/z + \cdots$. This can be viewed as a check of the correct normalization of the eigenvalue density distribution.

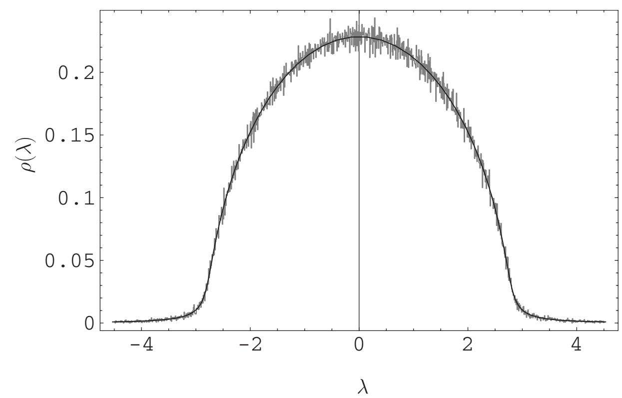

FIG. 1. Theoretical (black) and numerical (gray) eigenvalue distributions for $\mu = 1.95$.

F. Numerical comparison

In this section we show perfect agreement between the theoretical analysis for the eigenvalue distribution (27) and the numerically generated eigenvalue distribution. The latter is obtained by diagonalizing $N \times N$ random Lévy matrices sampled using the measure (1). The former was generated by calculating numerically $g(z)$ as detailed above. We performed numerically the inverse Cauchy transform (27). It is important to note that the integral transforms entering into the definition of the eigenvalue density $\rho(\lambda)$ in Eq. (27) converge slowly for $z \to \infty$. For that, we have used the asymptotic expansion of $g(z)$ to perform the large-z part of the integrals.

As noted above, all the above analytical results were obtained using a specific choice of the scale factor (16). The comparison with the numerically generated eigenvalue distribution generated using $L_\mu^{1,0}$ distribution requires rescaling through $\lambda \to \phi\lambda$ and $\rho(\lambda) \to \rho(\lambda)/\phi$ with

$$\phi = \left(\frac{\Gamma(1+\mu)\cos\left(\frac{\pi\mu}{4}\right)}{\Gamma\left(1+\frac{\mu}{2}\right)} \right)^{1/\mu}. \tag{45}$$

Below we show a sequence of results for $\mu = 1.95$, 1.75, 1.50, 1.25, and 1.00 with this rescaling (see Figs. 1–5). The comparison is for high statistics 400×400 samples (gray). We have checked that the convergence is good already for 100×100 samples, with no significant difference between $N = 100$, 200, and 400. The numerical results are also not sensitive to the choice $\beta \neq 0$. The agreement between the results following from the integral equations and the numerically generated spectra is perfect. This is true even for $\mu = 1$, where in principle the arguments used in the derivation may not be valid.

G. Numerical observation

In the mean-field approximation [7] one can assume that there are no correlations between large eigenvalues of the

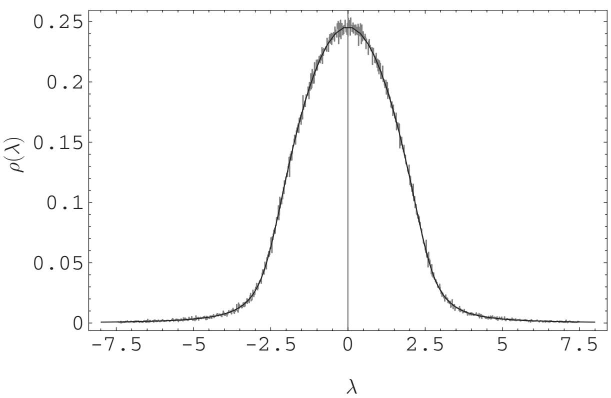

FIG. 2. Theoretical (black) and numerical (gray) eigenvalue distribution for $\mu=1.75$.

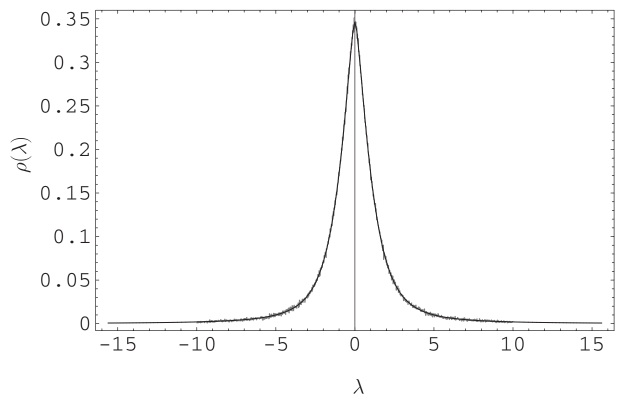

FIG. 4. Theoretical (black) and numerical (gray) eigenvalue distribution for $\mu=1.25$.

Wigner-Lévy random matrix. In this case the eigenvalue density takes the form

$$\hat{\rho}_\mu(\lambda) = L_{\mu/2}^{C(\lambda),\beta(\lambda)}(\lambda). \tag{46}$$

It is natural to ask how good this mean-field approximation is. This can be done by comparing the mean-field eigenvalue distribution $\hat{\rho}(\lambda)$ [Eq. (46)] to the eigenvalue distribution $\rho(\lambda)$ calculated by the inverse Hilbert transform (27) of the resolvent $g(z)$ [Eq. (25)] as we did in the previous section. We made this comparison numerically. The result of this numerical experiment was that within the numerical accuracy which we achieved the two curves representing $\hat{\rho}(\lambda)$ and $\rho(\lambda)$ lay on top of each other in the whole studied range of λ. Since our numerical codes are written in MATHEMATICA we could push the numerical accuracy very far, being only limited by the execution time of the code. We have not seen any sign of deviation between the shapes of the two curves [12]. This provides us with strong numerical evidence that the mean-field argument [7] gives an exact result but so far we have not managed to prove it. The value of the eigenvalue density $\hat{\rho}_\mu(0)$ for $\lambda=0$ can be calculated analytically for the mean-field density (46). Rescaling the density $\hat{\rho}_\mu(\lambda) \to \hat{\rho}_\mu(\phi\lambda)/\phi$ by the factor ϕ [Eq. (45)] we eventually obtain

$$\hat{\rho}_\mu(0) = \frac{\Gamma(1+2/\mu)}{\pi}\left(\frac{\Gamma(1+\mu/2)^2}{\Gamma(1+\mu)}\right)^{1/\mu}. \tag{47}$$

We draw this function in Fig. 6. In the same figure we also show points representing numerically evaluated values of the corresponding density $\rho_\mu(0)$ [Eq. (27)] at some values of μ. Within the numerical accuracy $\rho_\mu(0)$ and $\hat{\rho}_\mu(0)$ assume the same values.

The physical meaning of the mean-field argument is that indeed one can think of the large eigenvalues as independent of each other. A similar observation has been made recently [13]. The mathematical meaning of the mean-field argument is more complex as we shall discuss below.

Let us for brevity denote the function $L_{\mu/2}^{C(z),\beta(z)}(x)$, which is a function of two real arguments, by $f(z,x) \equiv L_{\mu/2}^{C(z),\beta(z)}(x)$. Equation (25) can be now written as

$$g(z) = \int_{-\infty}^{\infty} dx \frac{f(z,x)}{z-x} \tag{48}$$

and Eq. (27) as

$$\rho(\lambda) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} dz \frac{g(z)}{z-\lambda}. \tag{49}$$

When we insert Eq. (48) into the last equation we have

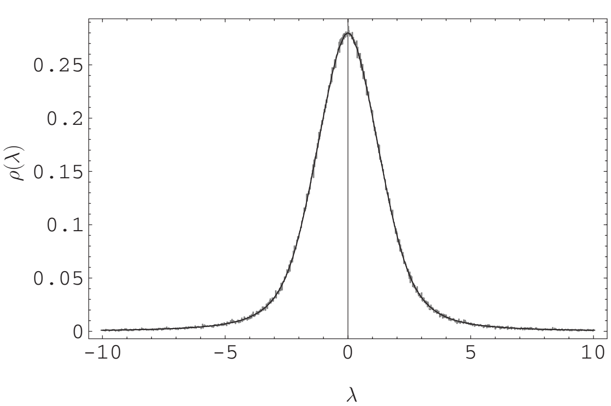

FIG. 3. Theoretical (black) and numerical (gray) eigenvalue distribution for $\mu=1.50$.

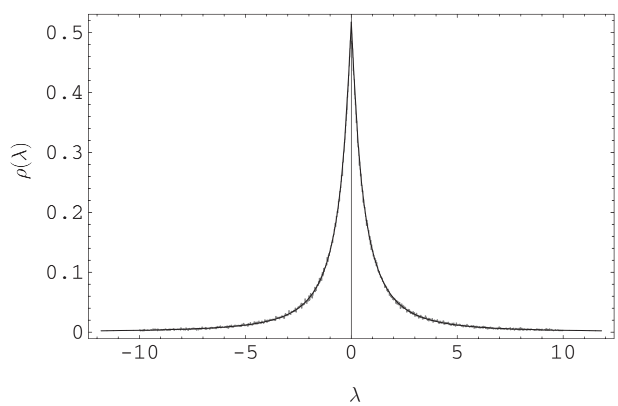

FIG. 5. Theoretical (black) and numerical (gray) eigenvalue distribution for $\mu=1.00$.

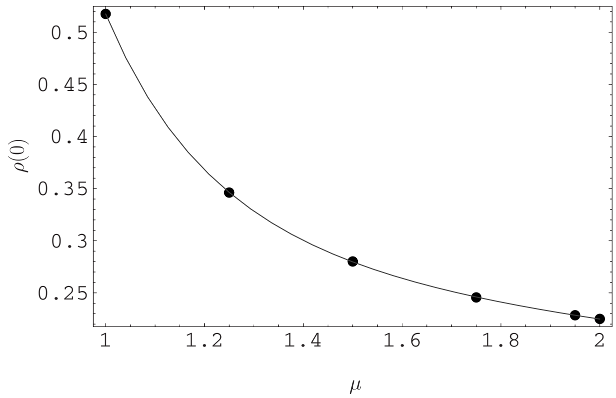

FIG. 6. The line represents the function $\hat{\rho}_\mu(0)$ [Eq. (47)] while the circles represent the values of $\rho_\mu(0)$ computed numerically for μ=1.0, 1.25, 1.5, 1.75, 1.95, and 2.0 from Eq. (27).

$$\rho(\lambda) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dx \frac{f(z,x)}{(z-\lambda)(z-x)}. \quad (50)$$

A question is when this exact expression for $\rho(\lambda)$ is equal to the mean-field solution: $\hat{\rho}(\lambda)=f(\lambda,\lambda)$. Recall the Poincaré-Bertrand theorem. It tells us that the following equation holds:

$$f(\lambda,\lambda) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dx \frac{f(z,x)}{(z-\lambda)(z-x)} - \frac{1}{\pi^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dz \frac{f(z,x)}{(z-\lambda)(z-x)}. \quad (51)$$

We see that the density $\rho(\lambda)$ is given by the mean-field result: $\rho(\lambda)=\hat{\rho}(\lambda)\equiv f(\lambda,\lambda)$ if the second term on the right-hand side of Eq. (51) vanishes. Unfortunately we have not managed to show that this is really the case for $f(z,x) = L_{\mu/2}^{C(z),\beta(z)}(x)$. One can, however, trivially observe that it would be the case if $f(z,x)$ had the following form: $f(z,x) = f(x,x) = L_{\mu/2}^{C(x),\beta(x)}(x)$, and probably also if $f(z,x)$ were a slowly varying function of z for z close to x, in which case the integral (48) would pick up only the contribution from $f(x,x)$ leading to Eq. (46).

III. FREE RANDOM LÉVY MATRICES

A. Rotationally invariant measure

Clearly Wigner-Lévy matrices are not rotationally invariant. In this section we shall discuss orthogonally (or unitary) invariant ensembles of Lévy matrices. It can be shown that maximally random measures for such matrices have the form [9,14]:

$$d\mu_{FR}(H) = \prod_{i\leq j} dH_{ij} e^{-N\,\mathrm{Tr}\,V(H)}. \quad (52)$$

We shall be interested only in potentials which have tails which lead to eigenvalue distributions (spectral densities) with heavy tails $\rho(\lambda)\sim\lambda^{-1-\mu}$ belonging to the Lévy domain of attraction. A generic form of $V(\lambda)$ at asymptotic eigenvalues λ is in this case

$$V(\lambda) = \ln\lambda^2 + O(1/\lambda^\mu). \quad (53)$$

In general the potential does not have to be an analytic function. We shall be interested here only in stable ensembles in the sense that the spectral measure (52) for the convolution of two independent and identical ensembles has the same form as the measure of the individual ensembles. In other words, the spectral measure for a matrix constructed as a sum of two independent matrices taken from the ensemble has exactly the same spectral measure (eigenvalue density) modulo linear transformations.

It turns out that one can classify all the stable spectral measures thanks to the relation of the problem to free probability calculus. The matrix ensemble (52) is in the large N limit a realization of free random variables [9], so one can use theorems developed in free random probability [8]. In particular we can use the fact that in free probability theory stable laws are classified. They actually parallel stable laws (3) of classical probability theory. In free probability the analog of the *logarithm* of the characteristic function (3) is the R-transform, introduced by Voiculescu [15]. The R-transform linearizes the matrix convolution, generating spectral cumulants, which are additive under convolution.

B. Stable laws in free probability

The remarkable achievement by Bercovici and Voiculescu [8] is an explicit derivation of all R-transforms defined by the equation $R[G(z)]=z-1/G(z)$, where $G(z)$ is the resolvent for all free stable distributions. We just note that $R[G(z)]$ is a sort of self-energy for rotationally symmetric FRL ensembles which is the analog of Eq. (13) which we previously defined for Wigner-Lévy ensembles. It is self-averaging and additive.

For stable laws $R(z)$ is known. It can have either the trivial form $R(z)=a$ or

$$R(z) = bz^{\mu-1}, \quad (54)$$

where $0<\mu<2$, b is a parameter which can be related to the stability index μ, the asymmetry parameter β, and the range C known from the corresponding stable laws (3) of classical probability [8,16]

$$b = \begin{cases} Ce^{i(\mu/2-1)(1+\beta)\pi} & \text{for } 1<\mu<2 \\ Ce^{i[\pi+\mu/2(1+\beta)\pi]} & \text{for } 0<\mu<1. \end{cases}$$

In the marginal case: $\mu=1$, $R(z)$ reads

$$R(z) = -iC(1+\beta) - \frac{2\beta C}{\pi}\ln Cz. \quad (55)$$

The branch cut structure of $R(z)$ is chosen in such a way that the upper complex half plane is mapped to itself. Recalling that $R=z-1/G$ in the large N limit, one finds that for the trivial case $R(z)=0$, the resolvent $G(z)=z^{-1}$, and the spectral distribution is a Dirac delta, $\rho(\lambda)=\delta(\lambda)$. Otherwise, on the upper half-plane, the resolvent fulfills an algebraic equation

$$bG^\mu(z) - zG(z) + 1 = 0, \quad (56)$$

or in the marginal case ($\mu=1$):

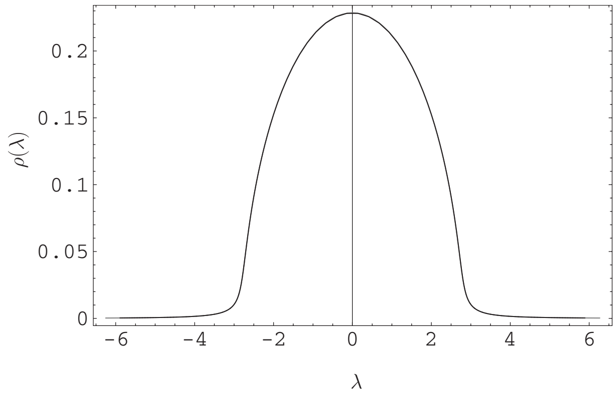

FIG. 7. WL (black) versus FRL (gray) for $\mu=1.95$.

$$[z+iC(1+\beta)]G(z)+\frac{2\beta C}{\pi}G(z)\ln CG(z)-1=0. \quad (57)$$

On the lower half-plane $G(\bar{z})=\bar{G}(z)$ [8].

The equation for the resolvent (56) has explicit solutions only for the following values: $\mu=1/4$, $1/3$, $1/2$, $2/3$, $3/4$, $4/3$, $3/2$, and 2. In all other cases the equation is transcendental and one has to apply numerical procedures to unravel the spectral distribution. Again the form of the potential generating stable free Lévy ensembles is highly nontrivial and is only known in a few cases [9]. We refer to [9] for further references and discussions.

C. Comparison of free Lévy and Wigner-Lévy spectra

We present in Figs. 7–11 several comparisons between the free random Lévy spectra (FRL) following from the solution to the transcendental equation (gray) and the random Lévy spectra (BC) obtained by solving the coupled integral equations (black), for zero asymmetry ($\beta=0$) and different tail indices μ. The FRL spectra are normalized to agree with the BC spectra in the tails of the distributions. We recall that the FRL spectra asymptote $\rho(\lambda)\approx\sin(\pi\mu/2)/\pi$. The comparison in bulk shows that the spectra are similar, in particular close to the Gaussian limit $\mu=2$, where both approaches become equivalent. For smaller μ there are differences.

WL and FRL matrices represent two types of random matrices spectrally stable under the matrix addition. For the WL matrices it follows from the measure, since each matrix element is generated from a stable Lévy distribution and therefore the sum of $\mathcal{N}$ WL matrices, scaled by $1/\mathcal{N}^{1/\mu}$, is equivalent to the original WL ensemble. The important point is that the WL measure is not symmetric while the FRL one is.

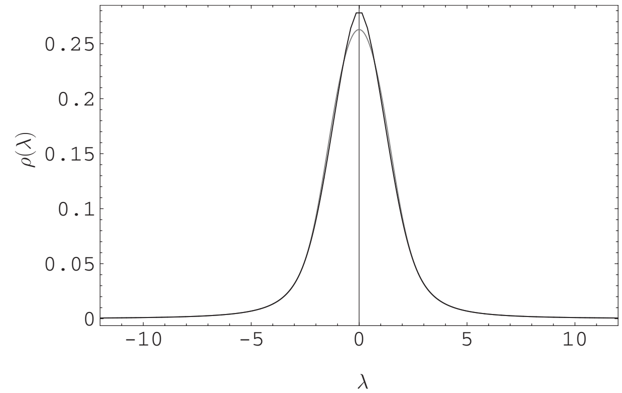

FIG. 9. WL (black) versus FRL (gray) for $\mu=1.50$.

IV. SPECTRAL STABILITY AND MAXIMAL ENTROPY PRINCIPLE

The matrix ensembles discussed in this paper are stable with respect to matrix addition in the sense that the eigenvalue distribution for the matrix constructed as a sum of two independent matrices from the original ensemble $H=H_1+H_2$ is identical as the original one up to a trivial rescaling. Wigner-Lévy matrices are obviously stable, since the probability distribution for individual matrix elements of the sum $H_{ij}=H_{1,ij}+H_{2,ij}$ is stable. A sum of two Wigner-Lévy matrices is again a Wigner-Lévy matrix. The Wigner-Lévy matrices are not rotatationally invariant. This means in particular that the eigenvalue distribution itself does not provide the whole information about the underlying matrix ensemble. Indeed, if O is a fixed orthogonal matrix, and H is a Wigner-Lévy matrix, then the matrix OHO^T is not a Wigner-Lévy matrix anymore but it has exactly the same eigenvalue distribution as H. In other words, an ensemble of Wigner matrices is not maximally random among ensembles with the same eigenvalue distribution. One expects that a maximally random ensemble with the given spectral properties should

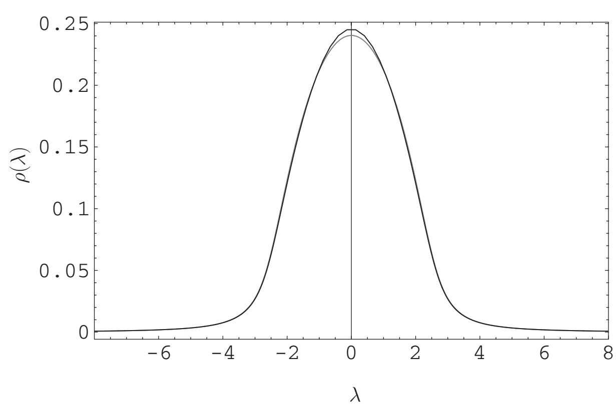

FIG. 8. WL (black) versus FRL (gray) for $\mu=1.75$.

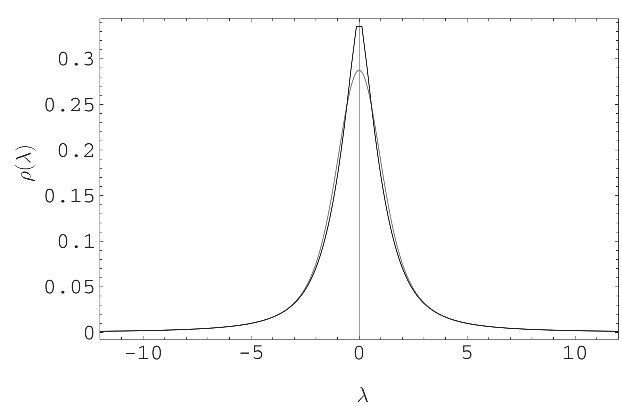

FIG. 10. WL (black) versus FRL (gray) for $\mu=1.25$

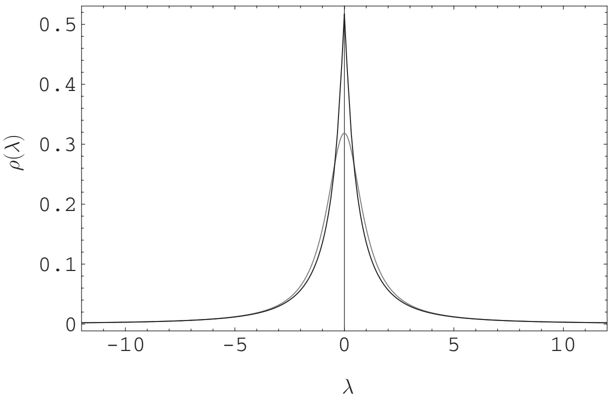

FIG. 11. WL (black) versus FRL (gray) for $\mu=1.00$.

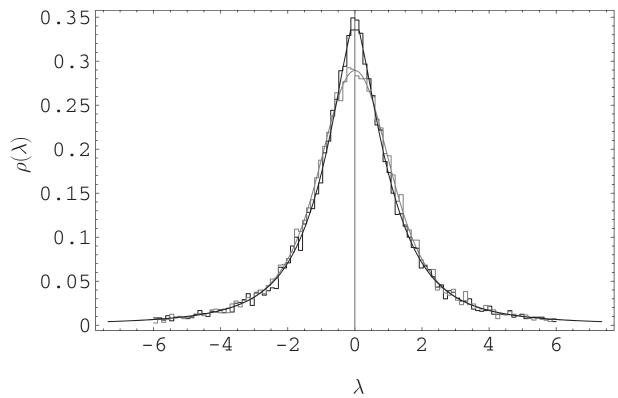

FIG. 13. WL (black) versus FRL (gray) stability for $\mu=1.25$.

be rotationally invariant. In this case one also expects that the stability holds not only for the sum $H=H_1+H_2$ but also for the sum of relatively rotated matrices: $H=H_1+OH_2O^T$, where O is an arbitrary orthogonal matrix. It can be shown [14] that the ensemble of random matrices which maximizes randomness (Shannon's entropy) for a given spectral density has the probability measure exactly of the form (52) as discussed here.

Stable laws are important because they define domains of attractions. For example, if one thinks of a matrix addition one expects that a sum of many independent identically distributed random matrices $H=H_1+\cdots+H_n$ should for $n\to\infty$ become a random matrix from a stable ensemble.

Maximally random spectrally stable ensembles which we discussed in the section on free random matrices play a special role since they can serve as an attraction point for the sums of IID rotationally invariant matrices. Moreover, one expects that even for not rotationally invariant random matrices H_i, the sums of the form $B=O_1H_1O_1^T+\cdots+O_nH_nO_n^T$ where O_i are random orthogonal matrices, will for large n generate a maximally random matrix B from a spectrally stable ensemble. In this spirit one can expect that if ones adds many randomly rotated Wigner-Lévy matrices,

$$B=\frac{1}{\mathcal{N}^{1/\mu}}\sum_i^{\mathcal{N}} O_iA_iO_i^T, \qquad (58)$$

that for $\mathcal{N}\to\infty$ the matrices B should become rotationally invariant, maximally random with a distribution governed by the FRL symmetric distribution. In Figs. 12–14 we show that this is indeed the case. The plots illustrate the two types of stability discussed above. In each case we generate $N=100$ WL matrices and combine $\mathcal{N}=100$ of them either as a simple sum (black) or a rotated sum (gray) with the appropriate scale factor. The plots represent the numerically measured spectra for the two cases. We present results for $\mu=1.5$, 1.25, and 1, which all show that a simple sum reproduces the BC result, while the rotated sum reproduces the symmetric FRL distribution.

V. CONCLUSIONS

We have given a detailed analysis of the macroscopic limit of two distinct random matrix theories based on Lévy type ensembles. The first one was put forward by Bouchaud and Cizeau [7] and uses a nonsymmetric measure under the orthogonal group, and the second one was suggested by us [9] and uses a symmetric measure.

After correcting the original analysis in [7], in particular our formulas (23) and (21) replace (10b) and (12b) in [7], and their Eq. (15) is replaced by the pair of "dispersion relations" (22) and (27), we found perfect agreement between the analytical and numerical spectra. The WL measure is easy to implement numerically for arbitrary asymmetry parameter β in the Lévy distributions. The spectrum of WL matrices does not depend on β and remains symmetric and universal, depending only on μ.

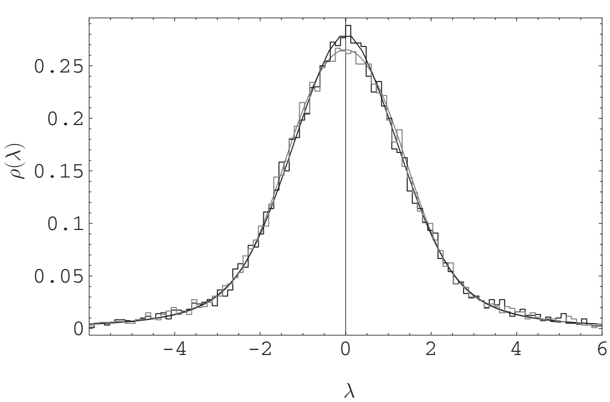

FIG. 12. WL (black) versus FRL (gray) stability for $\mu=1.5$

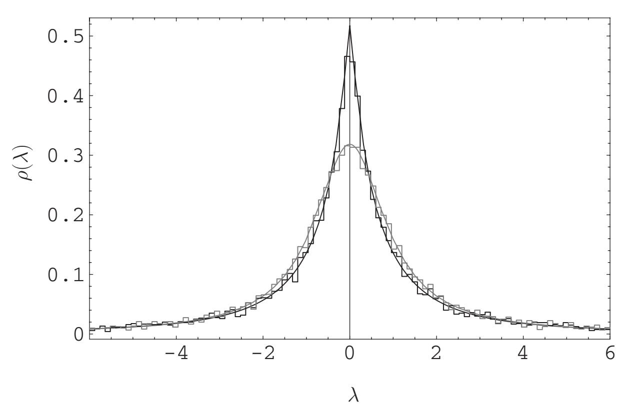

FIG. 14. WL (black) versus FRL (gray) stability for $\mu=1$.

We have also shown that the spectra generated analytically for symmetric FRL matrices are similar to the ones generated from WL matrices. Unlike the WL ensemble, the FRL ensemble allows for both symmetric and asymmetric Lévy distributions. Both ensembles are equally useful for addressing issues of recent interest [17,18].

Let us finish the paper with two remarks.

(1) The application of free probability calculus to asymptotically free matrix realizations allows one to derive spectral density of the matrices from the underlying matrix ensemble but it does not tell one how to calculate eigenvalue correlation functions, or joint probabilities for many eigenvalues. Actually different matrix realizations of free random variables may have a completely different structure of eigenvalue correlations even if they are realizations of the same free random variables. To fix correlations or joint probabilities for two or more eigenvalues, one has to introduce the concept of higher order freeness [19]. We think, however, that if one imposes on a matrix realization of free random variables an additional requirement that it has to be maximally random in the sense of maximizing Shannon's entropy [14], then this additional requirement automatically fixes the probability measure (52) for the ensemble and thus also all multieigenvalues correlations.

(2) Large eigenvalues behave differently for Wigner-Lévy and maximally random free random Lévy matrices discussed in this paper. As pointed out recently [13], the largest eigenvalues fluctuate independently for a Wigner-Lévy ensemble, a little bit like in the mean-field argument [7] mentioned before, while for the maximally random matrix ensemble (52) even large eigenvalues are correlated [9].

ACKNOWLEDGMENTS

We thank Jean-Philippe Bouchaud for discussions which prompted us to revisit and compare the two approaches presented in this paper. This work was partially supported by the Polish State Committee for Scientific Research (KBN) Grants No. 2P03B 08225 and No. 1P03B-04029 (2005-2008) (M.A.N., J.J., and Z.B.), Marie Curie TOK programme COCOS-Correlations in Complex Systems (Contract No. MTKD-CT-2004-517186) (M.A.N., J.J., Z.B, and G.P.), the National Office for Research and Technology Grant No. RET14/2005 (GP), and by US-DOE Grants No. DE-FG02-88ER40388 and No. DE-FG03-97ER4014 (I.Z.).

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